University of Bergen, Department of Physics
Scientific/Technical Report No.1995-06
ISSN 0803-2696

MZ-TH–95-13

hep-ph/9504432

April 1995

Two-loop vacuum diagrams and tensor decomposition [1]

A. I. Davydychev[a,2] and J. B. Tausk[b,3]

[a] *Department of Physics, University of Bergen,*
Allégaten 55, N-5007 Bergen, Norway

[b] *Institut für Physik, Johannes Gutenberg Universität,*
Staudinger Weg 7, D-55099 Mainz, Germany

Abstract

General algorithms for tensor reduction of two-loop massive vacuum diagrams are discussed. Some explicit useful formulae are presented.

[1] To appear in Proceedings of the AIHENP-95 conference, Pisa, April 1995 (World Scientific, Singapore, 1995)

[2] On leave from Institute for Nuclear Physics, Moscow State University, 119899, Moscow, Russia.
E-mail address: davyd@vsfys1.fi.uib.no

[3] E-mail address: tausk@vipmzw.physik.uni-mainz.de

1. General Remarks

In the talk given by one of the authors (A. D.) at the AIHENP-95 Conference in Pisa, some recent (i.e. since AIHENP-93) developments in two-loop calculations were reviewed. The main results discussed in the talk were published [1, 2, 3] and were obtained in collaboration with N.I. Ussyukina [1], F.A. Berends and V.A. Smirnov [2], and the other author of this paper, J.B. T. [2, 3].

In ref. [1] (see also in ref. [4]), some new results for two-loop three-point diagrams with massless internal particles were obtained, including the diagrams with irreducible numerators. The paper [2] continues the project started in refs. [5, 6] and provides explicit results for the small momentum expansion of two-loop self-energy diagrams for all cases when the smallest physical threshold is zero ("zero-threshold expansion"). To do this, general results on asymptotic expansions of Feynman diagrams were applied (see e.g. in the reviews [7] and references therein). In the paper [3], we have established a useful connection between two-loop massive vacuum diagrams and off-shell massless triangle diagrams. This connection is valid for all values of the space-time dimension n and masses m_i (which are related to the external momenta squared of the triangle diagram via $m_i^2 = p_i^2$, $i = 1, 2, 3$). As a corollary, the analytic result for the ε-part ($\varepsilon \equiv (4-n)/2$) of two-loop vacuum diagram was obtained in terms of trilogarithms. For the case of equal masses, a new transcendental constant (related to log-sine integral $\mathrm{Ls}_3(\frac{2}{3}\pi)$) is shown to appear.

Since it is impossible to give a detailed description of all these results in a short contribution, and also because the main results have been already published, we decided to concentrate ourselves on an important particular problem related to two-loop vacuum diagrams, namely the problem of tensor decomposition.

Tensor reduction of Feynman integrals containing loop momenta with uncontracted Lorentz indices in the numerator is very important for various realistic calculations in the Standard Model (and beyond). For one-loop diagrams with different numbers of external lines, several approaches and algorithms were developed [8]. For two-loop vacuum and self-energy diagrams, the problem was considered e.g. in refs. [9, 10], whilst the three-point two-loop case is more complicated and requires results for the integrals with irreducible numerators [1].

The problem of finding general algorithms for the tensor decomposition of two-loop vacuum diagrams is interesting because it is connected with the calculation of coefficients of the small momentum expansion [5], also for the three-point case [11]. In the three-point case, the problem is more tricky (even for scalar integrals), because we have two independent external momenta to contract with. Some relevant formulae for cases when the numerator is contracted with one or two external vectors can be found in refs. [6, 12]. We also note that the general result for a special case (when two masses are equal and the third is zero) was presented in ref. [10].

2. Tensor Decomposition

In this paper, we shall use the following notation:

$$I\,[\text{something}] \equiv \int\!\!\int \mathrm{d}^n p \; \mathrm{d}^n q \;\; \{\text{something}\} \;\; F\left(p^2, q^2, (pq)\right), \tag{1}$$

and we are interested in expressing the integrals

$$I\left[p_{\mu_1} \ldots p_{\mu_{N_1}} \; q_{\sigma_1} \ldots q_{\sigma_{N_2}}\right] \tag{2}$$

in terms of scalar integrals. In eq. (1), $F\left(p^2, q^2, (pq)\right)$ is an arbitrary scalar function depending on Lorentz invariants of the loop momenta p and q. In the special case when this function does not depend on (pq) we shall write it as $F\left(p^2, q^2\right)$. Usually, this function is a product of propagators,

$$\left(p^2 - m_1^2\right)^{-\nu_1} \left(q^2 - m_2^2\right)^{-\nu_2} \left((p-q)^2 - m_3^2\right)^{-\nu_3} \quad \text{or} \quad \left(p^2 - m_1^2\right)^{-\nu_1} \left(q^2 - m_2^2\right)^{-\nu_2} \;, \tag{3}$$

but all formulae we are going to discuss are valid for arbitrary scalar functions.

Some explicit results for the cases when the integral (2) was contracted with the external vector (momentum) k were given in ref. [6] (eqs. (B.10) and (B.9)). They are also valid when the integrands (3) are replaced by $F\left(p^2, q^2, (pq)\right)$ (in eq. (B.10)) or $F\left(p^2, q^2\right)$ (in eq. (B.9)), respectively. Moreover, we note that eq. (B.9) can be generalized to the case with two external momenta k_1 and k_2,

$$\begin{aligned}
&\int\!\!\int \mathrm{d}^n p \; \mathrm{d}^n q \; F\left(p^2, q^2\right) \; [2(k_1 p)]^{N_1} \; [2(k_2 q)]^{N_2} \; [2(pq)]^{N_3} \Bigg|_{\substack{N_1+N_3-\text{ even}\\ N_2+N_3-\text{ even}}} \\
&\qquad = \frac{N_1! \; N_2! \; N_3!}{(n/2)_{(N_1+N_2)/2} \; (n/2)_{(N_1+N_3)/2} \; (n/2)_{(N_2+N_3)/2}} \\
&\qquad\qquad \times \sum_{\substack{2j_1+j_3=N_1\\ 2j_2+j_3=N_2}} \frac{(k_1^2)^{j_1} (k_2^2)^{j_2} [2(k_1 k_2)]^{j_3}}{j_1! \; j_2! \; j_3!} \; \frac{(n/2)_{(N_1+N_2+N_3-j_3)/2}}{((N_3 - j_3)/2)!} \\
&\qquad\qquad \times \int\!\!\int \mathrm{d}^n p \; \mathrm{d}^n q \; F\left(p^2, q^2\right) \; (p^2)^{(N_1+N_3)/2} \; (q^2)^{(N_2+N_3)/2} \;,
\end{aligned} \tag{4}$$

where $(a)_j \equiv \Gamma(a+j)/\Gamma(a)$ denotes the Pochhammer symbol. If $(N_1 + N_3)$ or $(N_2 + N_3)$ is odd, the integral on the l.h.s. is equal to zero.

Since we have no external momenta in (2), only tensor structures constructed of metric tensors may be involved in the tensor decomposition. Therefore, (2) vanishes if $N \equiv N_1 + N_2$ is odd. For even N, the suitable tensor structures should be symmetric with respect to two subsets of indices, $(\mu_1, \ldots, \mu_{N_1})$ and $(\sigma_1, \ldots, \sigma_{N_2})$. All independent tensor structures can be constructed in the following way. Let us take a product of j_1 metric tensors $g_{\mu_i \mu_k}$, j_2 tensors $g_{\sigma_i \sigma_k}$ and j_3 tensors $g_{\mu_i \sigma_k}$ (we should remember that the conditions $2j_1 + j_3 = N_1$ and $2j_2 + j_3 = N_2$ should be satisfied). Now, let us consider all possible permutations of μ's and all possible permutations of σ's producing *distinct* products of

metric tensors, and take the sum of all these terms (with the coefficients equal to one). So, the obtained tensor structure is symmetrized in μ's and σ's, and it is

$$\{j_1, j_2, j_3\} \equiv g_{\mu_1\mu_2} \ldots g_{\mu_{2j_1-1}\mu_{2j_1}} g_{\sigma_1\sigma_2} \ldots g_{\sigma_{2j_2-1}\sigma_{2j_2}} g_{\mu_{2j_1+1}\sigma_{2j_2+1}} \ldots g_{\mu_{2j_1+j_3}\sigma_{2j_2+j_3}} + \text{permutations}. \tag{5}$$

For given N_1 and N_2, the number of independent tensor structures (5) is

$$T(N_1, N_2) = \min\left(\left[\tfrac{1}{2}N_1\right], \left[\tfrac{1}{2}N_2\right]\right) + 1, \tag{6}$$

where $\left[\frac{1}{2}N_i\right]$ is the integer part of $N_i/2$, and the number of terms on the r.h.s. of (5) is

$$c_{j_1j_2j_3} = \frac{N_1!\; N_2!}{2^{j_1+j_2}\; j_1!\; j_2!\; j_3!}. \tag{7}$$

Note that due to the conditions $2j_1 + j_3 = N_1$ and $2j_2 + j_3 = N_2$, at given N_1 and N_2 the tensor structures (5) are completely defined by one index, j_3.

So, the result for the integral (2) (for even $N_1 + N_2$) should look like

$$I\left[p_{\mu_1} \ldots p_{\mu_{N_1}}\; q_{\sigma_1} \ldots q_{\sigma_{N_2}}\right] = \sum_{\substack{2j_1+j_3=N_1 \\ 2j_2+j_3=N_2}} \{j_1, j_2, j_3\}\; I_{j_1j_2j_3}\;, \tag{8}$$

where $I_{j_1j_2j_3}$ are some scalar integrals which are to be found. A standard way to define $I_{j_1j_2j_3}$ is to consider all independent contractions of (8) (e.g. to contract it with each of the structures (5)). In such a way, one needs to solve a system of $T(N_1, N_2)$ (see eq. (6)) linear equations.

Due to power-counting reasons, the scalar integrals $I_{j_1j_2j_3}$ should be linear combinations of the integrals (1) with scalar numerators, carrying the same total powers of the loop momenta p and q. Therefore, we can re-write eq. (8) as

$$I\left[p_{\mu_1} \ldots p_{\mu_{N_1}}\; q_{\sigma_1} \ldots q_{\sigma_{N_2}}\right]$$

$$= \sum_{\substack{2j_1+j_3=N_1 \\ 2j_2+j_3=N_2}} \{j_1, j_2, j_3\} \sum_{\substack{2j_1'+j_3'=N_1 \\ 2j_2'+j_3'=N_2}} \phi_{j_1j_2j_3;j_1'j_2'j_3'}\; c_{j_1'j_2'j_3'}\; I\left[(p^2)^{j_1'}(q^2)^{j_2'}(pq)^{j_3'}\right], \tag{9}$$

where $c_{j_1j_2j_3}$ are defined in (7), while $\phi_{j_1j_2j_3;j_1'j_2'j_3'}$ can be considered as the elements of the "decomposition" matrix ($T \times T$, see eq. (6)). Contracting eq. (9) with all possible tensors (5), we get a "column" of $c_{j_1j_2j_3} I\left[(p^2)^{j_1}(q^2)^{j_2}(pq)^{j_3}\right]$ on the l.h.s. On the r.h.s., we get the "contraction" matrix χ with the elements defined as

$$\chi_{j_1j_2j_3;j_1'j_2'j_3'} = \text{contraction}\left(\{j_1, j_2, j_3\}, \{j_1', j_2', j_3'\}\right). \tag{10}$$

When speaking about matrices, we understand that only one of j-indices is relevant in each set (j_1, j_2, j_3) (e.g., j_3). For given N_1 and N_2, the values of j_3 can be either even (if

N_1 and N_2 are even) or odd (if N_1 and N_2 are odd). In order to use the matrix notation in ordinary form, we introduce the generalized index $j \equiv (j_1, j_2, j_3)$ (and $j' \equiv (j_1', j_2', j_3')$, etc.) which takes values from 1 to $T(N_1, N_2)$. In all "non-matrix" formulae, however, we would prefer to keep all the indices j_1, j_2, j_3 explicitly. So, in the matrix notation, a corollary of eqs. (9)–(10) is

$$\sum_{j''} \chi_{jj''}\, \phi_{j''j'} = \delta_{jj'} \; . \tag{11}$$

In other words, the decomposition matrix ϕ is nothing but the inverse contraction matrix, χ^{-1}. Since $\chi_{jj'}$ is symmetric (see (10)), $\phi_{jj'}$ is symmetric as well.

Thus, the problem is how to find all $\phi_{j_1j_2j_3;j_1'j_2'j_3'}$ for given N_1 and N_2. One way is to use recurrence relations. We note that the following simple formulae of contracting tensor structures (5) with respect to two indices can be derived:

$$g_{\mu_{N_1}\sigma_{N_2}}\{j_1,j_2,j_3\} = (n+2j_1+2j_2+j_3-1)\,\{j_1,j_2,j_3-1\} + (j_3+1)\,\{j_1-1,j_2-1,j_3+1\}\,, \tag{12}$$

$$g_{\mu_{N_1-1}\mu_{N_1}}\,\{j_1,j_2,j_3\} = (n+2j_1+2j_3-2)\,\{j_1-1,j_2,j_3\} + 2\,(j_2+1)\,\{j_1,j_2+1,j_3-2\} \tag{13}$$

(and also an analogous formula for contraction with $g_{\sigma_{N_2-1}\sigma_{N_2}}$). Using these contractions, the following recurrence relations for $\phi_{j_1j_2j_3;j_1'j_2'j_3'}$ can be obtained:

$$\begin{aligned} N_1N_2 \Big\{ (n+N_1+N_2-j_3'-1)\, \phi_{j_1j_2j_3;j_1'j_2'j_3'} + (j_3'-1)\, \phi_{j_1j_2j_3;j_1'+1,j_2'+1,j_3'-2} \Big\} \\ = j_3\, \phi_{j_1,j_2,j_3-1;j_1',j_2',j_3'-1}\;, \end{aligned} \tag{14}$$

$$\begin{aligned} N_1\,(N_1-1) \Big\{ (n+N_1+j_3'-2)\, \phi_{j_1j_2j_3;j_1'j_2'j_3'} + 2j_2'\, \phi_{j_1j_2j_3;j_1'-1,j_2'-1,j_3'+2} \Big\} \\ = 2\, j_1\, \phi_{j_1-1,j_2,j_3;j_1'-1,j_2',j_3'}\;, \end{aligned} \tag{15}$$

and also a relation similar to (15) but with interchanged indices, $1 \leftrightarrow 2$. In eqs. (12)–(15), it is understood that, whenever any of j's becomes negative, the corresponding tensors $\{j_1, j_2, j_3\}$ and ϕ's should be taken equal to zero.

If we increase N_1 and/or N_2 without changing the number of tensor structures (6), it is enough to have only relations (14)–(15) to express "higher" ϕ's in terms of "lower" ϕ's. If on this step one extra tensor structure appears (for example, when we go from odd N_1 and N_2 to (N_1+1) and (N_2+1)), we need one more relation between ϕ's. This extra relation can be obtained by contracting (9) with $k^{\mu_1}\ldots k^{\mu_{N_1}} k^{\sigma_1}\ldots k^{\sigma_{N_2}}$ and using eq. (B.10) of ref. [6],

$$\sum_{\substack{2j_1'+j_3'=N_1 \\ 2j_2'+j_3'=N_2}} \phi_{j_1j_2j_3;j_1'j_2'j_3'}\; c_{j_1'j_2'j_3'} = \frac{1}{2^{(N_1+N_2)/2}\;(n/2)_{(N_1+N_2)/2}}\;. \tag{16}$$

The values of the lowest ϕ's can be taken from the results at $N_2=0$ and $N_2=1$,

$$I\left[p_{\mu_1}\ldots p_{\mu_{N_1}}\right] = \frac{1}{2^{N_1/2}\;(n/2)_{N_1/2}} \left\{\frac{N_1}{2},0,0\right\}\; I\left[(p^2)^{N_1/2}\right]\;, \tag{17}$$

$$I\left[p_{\mu_1}\dots p_{\mu_{N_1}} q_{\sigma_1}\right] = \frac{1}{2^{(N_1+1)/2}\ (n/2)_{(N_1+1)/2}} \left\{\frac{N_1-1}{2}, 0, 1\right\}\ I\left[(p^2)^{(N_1-1)/2}(pq)\right]. \qquad (18)$$

Using the recurrence procedure, it is also possible to obtain some less trivial formulae, for $N_2 = 2$ and $N_2 = 3$ and arbitrary N_1 (such that $N_1 \geq N_2$):

$$\begin{aligned} I\left[p_{\mu_1}\dots p_{\mu_{N_1}} q_{\sigma_1} q_{\sigma_2}\right] &= \frac{1}{2^{(N_1+2)/2}\ (n/2)_{(N_1+2)/2}\ (n-1)} \\ &\times \left\{\left\{\frac{N_1}{2}, 1, 0\right\}\ I\left[(n+N_1-1)(p^2)^{N_1/2}q^2 - N_1(p^2)^{(N_1-2)/2}(pq)^2\right]\right. \\ &\left.+ \left\{\frac{N_1-2}{2}, 0, 2\right\}\ I\left[n(p^2)^{(N_1-2)/2}(pq)^2 - (p^2)^{N_1/2}q^2\right]\right\}, \qquad (19) \end{aligned}$$

$$\begin{aligned} I\left[p_{\mu_1}\dots p_{\mu_{N_1}} q_{\sigma_1} q_{\sigma_2} q_{\sigma_3}\right] &= \frac{1}{2^{(N_1+3)/2}\ (n/2)_{(N_1+3)/2}\ (n-1)} \\ &\times \left\{\left\{\frac{N_1-1}{2}, 1, 1\right\} I\left[(n+N_1-2)(p^2)^{(N_1-1)/2}q^2(pq) - (N_1-1)(p^2)^{(N_1-3)/2}(pq)^3\right]\right. \\ &\left.+ \left\{\frac{N_1-3}{2}, 0, 3\right\}\ I\left[(n+2)(p^2)^{(N_1-3)/2}(pq)^3 - 3(p^2)^{(N_1-1)/2}q^2(pq)\right]\right\}. \qquad (20) \end{aligned}$$

Thus, a recursive procedure of calculating $\phi_{j_1,j_2,j_3;j_1'j_2'j_3'}$ is constructed which provides a general algorithm to calculate the tensor integrals (2). It is difficult, however, to generalize eqs. (17)–(20) to the case of arbitrary N_2.

There is, however, another approach to this problem which does not involve using recurrence relations. Let us look again at the decomposition (9), remembering also how the integrals I are defined (1). Then, let us contract eq. (9) with $k_1^{\mu_1}\dots k_1^{\mu_{N_1}} k_2^{\sigma_1}\dots k_2^{\sigma_{N_2}}$, and multiply it by $(k_1k_2)^{N_3}\ F(k_1^2, k_2^2)$. Now, we can integrate the resulting expression over k_1 and k_2 (not over p and q!), using the formula (4). As a result of this integration at different values of N_3, we get $T(N_1, N_2)$ independent equations. Then, considering the scalar integrals $I\left[(p^2)^{j_1}(q^2)^{j_2}(pq)^{j_3}\right]$ as a basis, we get a set of relations for ϕ's. Finally, taking some linear combinations of these relations, we get a Kronecker symbol on the r.h.s. This means that we have found the elements of the matrix inverse to $\phi_{jj'}$ which is nothing but the contraction matrix $\chi_{jj'}$ (see eqs. (10)–(11)). The obtained expression is

$$\begin{aligned} \chi_{j_1j_2j_3;j_1'j_2'j_3'} = \frac{N_1!\ N_2!}{j_1!j_2!j_3!} \sum_{l=0}^{[j_3'/2]} & \frac{(-1)^l\ (j_3+j_3'-2l)!\ ((n+N_1+N_2-2l+2)/2)_l}{l!\ (j_3'-2l)!\ ((j_3+j_3'-2l)/2)!\ (n/2)_{(j_3+j_3'-2l)/2}} \\ & \times (n/2)_{(N_1+j_3'-2l)/2}\ (n/2)_{(N_2+j_3'-2l)/2}\,. \qquad (21) \end{aligned}$$

Thus, we have obtained an explicit general expression for the inverse of the decomposition matrix, $(\phi^{-1})_{jj'}$. For the cases of interest, the matrix (21) can be inverted by use of computer systems for analytical calculations. Although it would be nicer to get an analogous result for the "direct" matrix $\phi_{jj'}$, the presented approach can also be considered as a general solution to the problem of tensor decomposition of two-loop vacuum diagrams.

Acknowledgements

A. D. is grateful to K.G. Chetyrkin and O.V. Tarasov for useful discussions. A. D.'s research and participation in AIHENP-95 conference in Pisa were supported by the Research Council of Norway. J.B.T. was supported by the Graduiertenkolleg "Teilchenphysik" in Mainz.

References

[1] N.I. Ussyukina and A.I. Davydychev, *Phys. Lett.* B332 (1994) 159; B348 (1995) 503.

[2] F.A. Berends, A.I. Davydychev, V.A. Smirnov and J.B. Tausk, *Nucl. Phys.* B439 (1995) 536.

[3] A.I. Davydychev and J.B. Tausk, Mainz/Bergen preprint MZ-TH-95-14, Bergen-1995-07 (hep-ph/9504431).

[4] N.I. Ussyukina and A.I. Davydychev, *Phys. Lett.* B298 (1993) 363; B305 (1993) 136.

[5] A.I. Davydychev and J.B. Tausk, *Nucl. Phys.* B397 (1993) 123.

[6] A.I. Davydychev, V.A. Smirnov and J.B. Tausk, *Nucl. Phys.* B410 (1993) 325.

[7] V.A. Smirnov, *Renormalization and Asymptotic Expansions* (Birkhäuser, Basel, 1991);
K.G. Chetyrkin, Munich preprint MPI-PAE/PTh 13/91 (1991);
F.V. Tkachov, *Phys. Part. Nucl.* 25 (1994) 272.

[8] L.M. Brown and R.P. Feynman, *Phys. Rev.* 85 (1952) 231;
G. Passarino and M. Veltman, *Nucl. Phys.* B160 (1979) 151;
F.V. Tkachov, *Phys. Lett.* B100 (1981) 65;
K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys.* B192 (1981) 159;
R.G. Stuart, *Comput. Phys. Commun.* 48 (1988) 367; 85 (1995) 267;
R.G. Stuart and A. Gongora-T., *Comput. Phys. Commun.* 56 (1990) 337;
G.J. van Oldenborgh and J.A.M. Vermaseren, *Z. Phys.* C46 (1990) 425;
A.I. Davydychev, *Phys. Lett.* B263 (1991) 107;
Y. Ezawa e.a., *Comput. Phys. Commun.* 69 (1992) 15.

[9] G. Weiglein, R. Scharf and M. Böhm, *Nucl. Phys.* B416 (1994) 606;
D. Kreimer, *Mod. Phys. Lett.* A9 (1994) 1105.

[10] K.G. Chetyrkin, in: *New Computing Techniques in Physics Research III*, eds. K-H. Becks and D. Perret-Gallix (World Scientific, Singapore, 1994), p. 559.

[11] J. Fleischer and O.V. Tarasov, *Z. Phys.* C64 (1994) 413.

[12] J. Fleischer and O.V. Tarasov, Bielefeld preprint BI-TP–94-61 (hep-ph/9502257).